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An Asymptotic Result for the Degree of Approximation by Monotone Polynomials

VASANT A. UBHAYA

Department of Computer Science and Operations Research, Division of Mathematical Sciences, 300 Minard Hall, North Dakota State University, Fargo, North Dakota 58105, U.S.A.

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The problem considered is the approximation of a continuous function defined on an interval by polynomials which are monotone nondecreasing there. Upper and lower bounds on the degree of approximation as well as an asymptotic result are obtained.

1. INTRODUCTION

The problem considered is the approximation of a continuous real function defined on an interval by polynomials which are monotone nondecreasing there. Upper and lower bounds on the degree of approximation and an asymptotic result are established. The latter shows that when f is continuous but not nondecreasing, $E_n(f)$, the degree of approximation of f by monotone polynomials of degree at most n, converges to a positive number as $n \to \infty$ at a geometric rate. The complementary case when f is continuous and nondecreasing was investigated earlier in the literature.

Let P_n denote the class of all nondecreasing polynomials of degree at most n defined on a real interval $I = [a, b]$. Given a continuous function f on I, not necessarily nondecreasing, the problem of monotone polynomial approximation is to find a q_n in P_n such that $\|f - q_n\|$ minimizes $\|f - p_n\|$ for all p_n in P_n, where $\|\cdot\|$ is the uniform norm given by $\|f\| = \max\{|f(x)|: x \in I\}$. The number $E_n(f)$, defined by

$$E_n(f) = \|f - q_n\| = \min\{\|f - p_n\|: p_n \in P_n\},$$

is known as the degree of approximation of f by polynomials in the class P_n.

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If f is any bounded function on I, then, analogous to its well-known modulus of continuity $\omega(f, \cdot)$, define two nonnegative functions $\underline{\mu}$ and $\bar{\mu}$ by

$$\underline{\mu}(\delta) = \underline{\mu}(f, \delta) = \sup\{(f(y) - f(x)): x, y \in I, 0 \leqslant y - x \leqslant \delta\}, \quad \delta \in [0, l],$$

$$\bar{\mu}(\delta) = \bar{\mu}(f, \delta) = \sup\{(f(x) - f(y)): x, y \in I, 0 \leqslant y - x \leqslant \delta\}, \quad \delta \in [0, l],$$

where $l = b - a$. The functions $\underline{\mu}(f, \cdot)$ and $\bar{\mu}(f, \cdot)$ are called the moduli of monotonicity, decreasing and increasing, respectively, of f [16]. Clearly, $\omega = \max(\underline{\mu}, \bar{\mu})$, and thus $\underline{\mu}$ and $\bar{\mu}$ give a decomposition of ω in this sense. Also, $\omega = \underline{\mu}$ if f is nondecreasing.

If f is continuous and nondecreasing, then analogous to the well-known Weierstrass Approximation Theorem, we have $E_n(f) \to 0$ as $n \to \infty$. A bound on $E_n(f)$, which was obtained by Lorentz and Zeller [7] for such an f, may be written in the form [16, p. 122]

$$E_n(f) \leqslant c_0 \underline{\mu}(f, l/(2n)), \tag{1.1}$$

where c_0 is an absolute constant. If, on the other hand, f is continuous but not nondecreasing, then it is shown in [16] that as $n \to \infty$,

$$\text{limit } E_n(f) = \tfrac{1}{2}\,\bar{\mu}(f, l) > 0.$$

Bounds on $E_n(f)$ were also established there. These investigations are pursued further in this article under the sole assumption of continuity of f. Our results in Section 3 show that for a fixed continuous f, which is not nondecreasing, and any fixed positive integer m, the nonnegative number $E_n(f) - \frac{1}{2}\,\bar{\mu}(f, l)$ is bounded above by

$$c_1(n+1)\,\rho^{m+1}\binom{n+1}{m+1}^{-1}$$

for all $n \geqslant m$, where ρ and c_1 are independent of n and m but dependent on f and $\rho \geqslant 2$. It is established asymptotically that, for such an f, $E_n(f) - \frac{1}{2}\,\bar{\mu}(f, l)$ does not exceed

$$c_2 n^{3/2}(\rho/(\rho+1))^n \tag{1.2}$$

as $n \to \infty$, where c_2 is dependent on f. The value of ρ is given in Section 3. Our results obviously complement result (1.1) of Lorentz and Zeller since the continuous function f belongs to complementary sets in the respective cases. The bound (1.2) clearly implies geometric convergence, viz.,

$$0 \leqslant E_n(f) - \tfrac{1}{2}\,\bar{\mu}(f, l) \leqslant A(\theta)\,\theta^n \tag{1.3}$$

as $n \to \infty$, for all θ satisfying $\rho/(\rho+1) < \theta < 1$ and some $A(\theta) > 0$.

Svedov [15] has obtained the following result for a continuous and non-decreasing f:

$$E_n(f) \leqslant c_3 \omega_2(f, l/n), \tag{1.4}$$

where ω_2 is the modulus of smoothness of order 2. This improves an earlier estimate of Lorentz [6, Theorem 12]. To compare (1.1), (1.3), and (1.4) we simply observe that $\omega_2 \leqslant 2\omega = 2\mu$ and $\omega_2(f, l/n)/\theta^n \to \infty$ as $n \to \infty$.

One of the earlier works on monotone polynomial appoximation is by Shisha [12], who obtained bounds on the degree of approximation under various differentiability conditions. (Tchebycheff considered the problem in 1873. See [15].) Subsequently several articles [1, 4, 6, 7, 10, 11, 15, 16] including surveys [3, 5] have appeared on the topic. Many of these articles impose conditions stronger than continuity on f. Some approximation problems with constraints appear in [3, 5, 14, 19, 20] and the references given there.

The problem of approximating a nonmonotone function by monotone polynomials or functions arises as a curve fitting or estimation problem. The initial data points $f(x)$, based on experimental observations, may be nonmonotone because they display certain random variations, but it is of interest to obtain a monotone fit based on the data. One example is to determine the failure rate of a complex system from observed failure data under the assumption that the failure rate is nondecreasing. As a result of random fluctuations, the probability of the initial data itself being monotone is small.

2. Preliminaries

In this section we obtain asymptotic bounds for $\binom{n}{k}$, when k increases with n in some manner specified in advance. These bounds will be used in the next section to obtain asymptotic results. We shall use the following well-known inequalities (see [2] or [8, p. 196]). Let n and k $(n > k)$ be natural numbers and let

$$Q(n, k) = \left(\frac{n}{2\pi k(n-k)}\right)^{1/2} \left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k}.$$

Then

$$Q(n, k) \exp\left(\frac{1}{12n + \frac{1}{4}} - \frac{1}{12k} - \frac{1}{12(n-k)}\right) < \binom{n}{k}, \tag{2.1}$$

and

$$\binom{n}{k} < Q(n,k)\exp\left(\frac{1}{12n} - \frac{1}{12k+\frac{1}{4}} - \frac{1}{12(n-k)+\frac{1}{4}}\right). \tag{2.2}$$

In this section all limits are taken as $n \to \infty$.

PROPOSITION 2.1. *Let $\langle r_n \rangle$ be a real sequence such that limit $r_n = r$, where $0 < r < 1$. Let $s \leqslant t$. For each n, let there exist a positive integer k_n satisfying*

$$nr_n + s \leqslant k_n \leqslant nr_n + t. \tag{2.3}$$

Define

$$G_n = \frac{(k_n/n)^{k_n}(1-k_n/n)^{n-k_n}}{(r_n)^{k_n}(1-r_n)^{n-k_n}}, \qquad n = 1, 2, \ldots \tag{2.4}$$

and

$$H_n = \frac{\binom{n}{k_n}^{-1}}{\sqrt{n}\,(r_n)^{k_n}(1-r_n)^{n-k_n}}, \qquad n = 1, 2, \ldots. \tag{2.5}$$

Then,

$$\exp(s-t) \leqslant \liminf G_n \leqslant \limsup G_n \leqslant \exp(-s+t) \tag{2.6}$$

and

$$\begin{aligned}&(2\pi r(1-r))^{1/2}\exp(s-t)\\ &\quad\leqslant \liminf H_n \leqslant \limsup H_n \leqslant (2\pi r(1-r))^{1/2}\exp(-s+t).\end{aligned} \tag{2.7}$$

Proof. We first establish (2.6). There are three cases to be considered, $0 \leqslant s \leqslant t$, $s < 0 \leqslant t$, and $s \leqslant t < 0$. First consider the case $s < 0 \leqslant t$. By hypothesis, there exists n_0 such that for all $n \geqslant n_0$ we have $0 < r_n < 1$, $1 \leqslant r_n n + s$, and $1 \leqslant (1-r_n)n - t$. By (2.3) we have, for $n \geqslant n_0$,

$$\left(1+\frac{s}{nr_n}\right)^{nr_n+t} \leqslant \frac{(k_n/n)^{k_n}}{(r_n)^{k_n}} \leqslant \left(1+\frac{t}{nr_n}\right)^{nr_n+t}.$$

Taking limits we obtain

$$\exp(s) \leqslant \liminf \frac{(k_n/n)^{k_n}}{(r_n)^{k_n}} \leqslant \limsup \frac{(k_n/n)^{k_n}}{(r_n)^{k_n}} \leqslant \exp(t). \tag{2.8}$$

Again (2.3) gives

$$(1-r_n)n-t \leqslant n-k_n \leqslant (1-r_n)n-s.$$

Using arguments as above, we have

$$\begin{aligned} \exp(-t) &\leqslant \liminf \frac{(1-k_n/n)^{n-k_n}}{(1-r_n)^{n-k_n}} \\ &\leqslant \limsup \frac{(1-k_n/n)^{n-k_n}}{(1-r_n)^{n-k_n}} \leqslant \exp(-s). \end{aligned} \tag{2.9}$$

Inequalities (2.8) and (2.9) give the required result (2.6) when $s<0\leqslant t$. The other two cases may be considered similarly. Thus (2.6) is established.

To establish (2.7), we observe that $n>k_n$ for all sufficiently large n. We substitute k_n for k in inequalities (2.1) and (2.2). Then using (2.4) and (2.5) we obtain the following for all large n:

$$\begin{aligned} &\exp\left(-\frac{1}{12n}+\frac{1}{12k_n+\frac{1}{4}}+\frac{1}{12(n-k_n)+\frac{1}{4}}\right) \\ &\quad < \left(2\pi\left(\frac{k_n}{n}\right)\left(1-\frac{k_n}{n}\right)\right)^{-1/2}(H_n/G_n) \\ &\quad < \exp\left(-\frac{1}{12n+\frac{1}{4}}+\frac{1}{12k_n}+\frac{1}{12(n-k_n)}\right). \end{aligned}$$

Since $k_n/n \to r$, we obtain

$$\operatorname{limit}(H_n/G_n)=(2\pi r(1-r))^{1/2}.$$

This together with (2.6) establishes (2.7). The proof is now complete.

3. Main Results

In this section we state and prove our main results.

We first introduce some notation and preliminary results. Let C denote the space of real continuous functions defined on the interval $I=[a,b]$ and K the convex cone of real nondecreasing functions on I. Recall that $l=b-a$. We observe that if $f\in C-K$, then $\bar{\mu}(f,l)>0$, and by [16, Sect. 4, Lemma 1], we have

$$\lambda=\sup\{\delta\in[0,l]: \omega(f,\delta)=\bar{\mu}(f,l)\}>0.$$

If Φ is the class of all Friedrichs mollifier functions ϕ with support in $[0, 1]$, we let

$$\Delta_k = \inf\left\{\int_0^1 |\phi^{(k)}(t)|\, dt \colon \phi \in \Phi\right\}, \qquad k = 1, 2, ...,$$

where $\phi^{(k)}$ denotes the kth derivative of ϕ [9]. The numbers Δ_k appear in the main results of [16] and are used in this section. It is shown in [17, 18] that

$$\Delta_k = k!\, 2^{2k-1}, \qquad k = 1, 2, ... \tag{3.1}$$

We derive our results from Theorems 2 and 3 of [16]. These theorems were in turn derived respectively from Theorems 5 and 1 of Shisha [12]. To obtain his Theorem 1, Shisha essentially used the well-known estimate of Farvad and Ahiezer-Krein expressed in the following form (see [16, Ref. 1 and 2]): If $E_n^*(f)$ denotes the degree of approximation of an f in C by polynomials of degree at most n, then

$$E_n^*(f) \leqslant (\pi l/4)^k \left(\prod_{j=0}^{k-1} (n+1-j)\right)^{-1} \|f^{(k)}\|, \tag{3.2}$$

where $f^{(k)}$ is the kth derivative of f. Sinwel [13] has improved this estimate to

$$E_n^*(f) \leqslant (\pi/2)(l/2)^k \left(\prod_{j=0}^{k-1} (n+1-j)\right)^{-1} \|f^{(k)}\|. \tag{3.3}$$

If we use (3.3) instead of (3.2) in Shisha's argument and then use his Theorem 1 thus modified in our earlier work [16], we obtain the following improved version of our Theorem 3 of [16]: If $f \in C - K$, then for every positive integer m and for all $n \geqslant m$,

$$0 \leqslant E_n(f) - \tfrac{1}{2}\bar{\mu}(f, l) \leqslant \theta(f, l, m)\left(\prod_{j=1}^{m} (n+1-j)\right)^{-1},$$

where

$$\theta(f, l, m) = 2\pi(\|f\| + \tfrac{1}{2}\bar{\mu}(f, l))\left(\frac{l}{2\lambda}\right)^{m+1} \Delta_{m+1}.$$

Substituting for the value of Δ_{m+1} from (3.1) we get

THEOREM 3.1. *If $f \in C-K$, then for every positive integer m and for all $n \geqslant m$,*

$$\begin{aligned} 0 &\leqslant E_n(f) - \tfrac{1}{2}\bar{\mu}(f, l) \\ &\leqslant \pi(\|f\| + \tfrac{1}{2}\bar{\mu}(f, l))\,\rho^{m+1}(n+1)\binom{n+1}{m+1}^{-1}, \end{aligned} \tag{3.4}$$

where $\rho = 2l/\lambda \geqslant 2$.

We now obtain our asymptotic result.

THEOREM 3.2. *If $f \in C-K$ then*

$$0 \leqslant E_n(f) - \tfrac{1}{2}\bar{\mu}(f, l) \leqslant c_2 n^{3/2}(\rho/(\rho+1))^n, \tag{3.5}$$

as $n \to \infty$ for some c_2 which depends upon f.

Proof. For $n \geqslant 1$, let

$$A_n(m) = \rho^{m+1}(n+1)\binom{n+1}{m+1}^{-1}, \qquad m = 0, 1, ..., n.$$

Note that $A_n(m)$, $1 \leqslant m \leqslant n$, contains all terms of the right side of (3.4) that involve m. Let m_n be the smallest positive integer which minimizes $A_n(m)$ subject to $m \in \{1, 2, ..., n\}$. We assert that if $n > 2\rho$ then $m_n \in [(n-2\rho)/(\rho+1), (n-\rho+1)/(\rho+1))$. To show this, we let $\Delta(m) = A_n(m) - A_n(m-1)$, $m = 1, 2, ..., n$. It is easy to verify that

$$\Delta(m) = (n!)^{-1}\,\rho^m m!\,(n-m)!(m(\rho+1) - (n-\rho+1)).$$

For convenience, let $q = (n-\rho+1)/(\rho+1)$. If $n > 2\rho$ then $q > 1$. In this case $\Delta(m) < 0$ for all $1 \leqslant m < q$ and $\Delta(m) \geqslant 0$ for all $q \leqslant m \leqslant n$. The assertion is thus established. Since $(n-\rho+1)/(\rho+1) = (n-2\rho)/(\rho+1) + 1$, the integer m_n is unique. Also $m_n/n \to 1/(\rho+1)$. Now we write

$$\frac{A_n(m_n)}{n^{3/2}(\rho/(\rho+1))^n} = \rho\left(\frac{m_n+1}{n}\right)\frac{\binom{n}{m_n}^{-1}}{\sqrt{n}\,(1/(\rho+1))^{m_n}(\rho/(\rho+1))^{n-m_n}}, \tag{3.6}$$

where, as was shown above,

$$n/(\rho+1) - 2\rho/(\rho+1) \leqslant m_n \leqslant n/(\rho+1) - (\rho-1)/(\rho+1).$$

We now apply Proposition 2.1 to (3.6) with $k_n = m_n$, $r = 1/(\rho+1)$,

$s=-2\rho/(\rho+1)$, and $t=-(\rho-1)/(\rho+1)$ to conclude that (3.6) is bounded above and below away from zero as $n \to \infty$. Since, by (3.4),

$$0 \leqslant E_n(f) - \tfrac{1}{2}\bar{\mu}(f, l) \leqslant dA_n(m_n)$$

for some constant d, the required result (3.5) is established.

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